Application of a Property of the Airy Function to Fiber Optics

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Abstract—The integral of the square of the Airy function from one of its zeros to infinity is equal to the square of the first derivative of the Airy function at the zero considered. Two important applications of this result to fiber optics are discussed.

The Airy function is involved in many problems of fiber optics. For example, waves guided along the curved boundary of a homogeneous dielectric [1] (whispering gallery modes [2]) or along the straight boundary of a medium with constant transverse gradient of refractive index [3], are described by Airy functions. We shall show that the normalized field at the dielectric boundary is given by a very simple expression because of a property of the Airy function that does not seem to be known. Knowledge of the normalized field is essential to evaluate the coupling strength and the bending loss of a mode.

The Airy function $\mathrm{Ai}(x)$ is a solution of the differential equation [4]

$$d^2 \mathrm{Ai}\,(x)/dx^2 = x \,\mathrm{Ai}\,(x). \tag{1}$$

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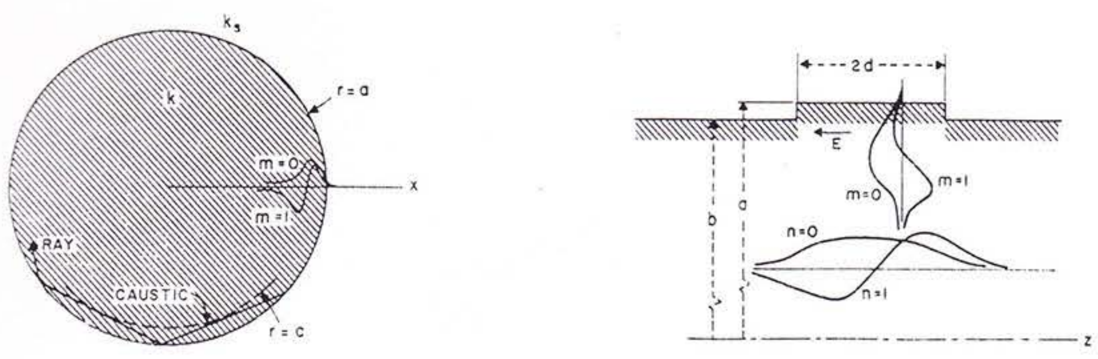

(a) (b)

Fig. 1. (a) Represents a cross section of the dielectric rod. The mode field has an oscillatory behavior between the caustic with radius c. and the rod boundary with radius a. The field decays exponentially in the surrounding medium. (b) The wave can be kept confined in the axial (z) direction by a slight reduction of the rod radius, from $r = a$ to $r = b$.

Let us evaluate the integral

$$I = \int_x^\infty \mathrm{Ai}^2(x)\, dx. \tag{2}$$

Integrating by parts, we have

$$I = x\, \mathrm{Ai}^2(x) \Big|_x^\infty - 2 \int_x^\infty x\, \mathrm{Ai}(x)[d\, \mathrm{Ai}(x)/dx]\, dx. \tag{3}$$

If we use the differential equation (1), (3) can be written

$$I = x\, \mathrm{Ai}^2(x) \Big|_x^\infty - 2 \int_x^\infty [d\, \mathrm{Ai}(x)/dx][d^2 \mathrm{Ai}(x)/dx^2]\, dx$$

$$= x\, \mathrm{Ai}^2(x) \Big|_x^\infty - [d\, \mathrm{Ai}(x)/dx]^2 \Big|_x^\infty$$

$$= -x\, \mathrm{Ai}^2(x) + [\mathrm{Ai}'(x)]^2 \tag{4}$$

where

$$\mathrm{Ai}'(x) \equiv d\, \mathrm{Ai}(x)/dx$$

because $\lim (x \to \infty)$ of $x\, \mathrm{Ai}^2(x) = 0$, and $\lim (x \to \infty)$ of $\mathrm{Ai}'(x) = 0$. If $x = x_a$ is a zero of the Airy function, the simple result

$$\int_{x_a}^\infty \mathrm{Ai}^2(x)\, dx = [\mathrm{Ai}'(x_a)]^2 \tag{5}$$

is obtained.

As a first example of application of (5), let us consider whispering gallery modes guided along the circular boundary of a dielectric rod, with radius a. The number of plane waves in the dielectric material is denoted k. The wavenumber of plane waves in the surrounding medium (or cladding) is denoted k_s. We assume that the ratio k/k_s is not very different from unity and make the scalar approximation.

The field of whispering gallery modes has the form

$$\psi(x) = \psi_0\, \mathrm{Ai}[\kappa(-x - a + c)], \qquad x < 0 \tag{6}$$

where Ψ_0 is a constant and

$$\kappa = 2^{1/3} k^{2/3} c^{-1/3} \tag{7}$$

as we can see by taking the asymptotic form of Bessel's functions. In (6) and (7), c denotes the caustic radius, a quantity that we shall define later, and $x \equiv r - a$ [see Fig. 1(a)]. The azimuthal wavenumber is equal to k at the caustic radius c, and therefore, it is equal to $(c/a)k$ at the rod radius a. The field outside the rod is approximately given by an exponential

$$\Psi(x) \approx \exp(-sx), \qquad x > 0$$

$$s = (k^2c^2/a^2 - k_s^2)^{1/2}. \tag{8}$$

Continuity of the field and of its first derivative at the rod boundary $r = a$ (or $x = 0$) requires that from (6) and (8)

$$\psi_0\, \mathrm{Ai}[\kappa(-a + c)] = 1$$

$$\psi_0 \kappa\, \mathrm{Ai}'[\kappa(-a + c)] = s. \tag{9}$$

The caustic radius c is now defined by (9).

The power of the mode is proportional to the integral of $k\psi^2(x)$ from $x = -\infty$ to the rod boundary, p is the integral of $k_s\psi^2(x)$ from the rod boundary to $x = +\infty$. Using (6), (8), (9), and the result in (4) we obtain

$$P = k \int_{-\infty}^0 \psi_0^2\, \mathrm{Ai}^2[\kappa(-x - a + c)]\, dx + k_s \int_0^\infty \exp(-2sx)\, dx$$

$$= ks^2/\kappa^3 + k(a - c) + k_s/2s. \tag{10}$$

For large values of the rod normalized frequency $F \equiv (k^2 - k_s^2)^{1/2} a$ (sometimes denoted V), and low-order modes, the wave clings tightly to the boundary ($c - a \ll a$) and the field at the boundary is very small compared with the field inside the rod in the annular region $c < r < a$. Thus, in the limit $F \to \infty$, the square of the normalized field $\hat{\psi}^2 = \psi^2/P$ at the rod boundary is obtained by neglecting the last two terms in (10), using (7) and the approximation $s \approx (k^2 - k_s^2)^{1/2}$. The result is

$$\hat{\psi}^2 \equiv \psi^2/P \approx \kappa^3/ks^2 = 2k/(k^2 - k_s^2)a. \tag{11}$$

It is remarkable that this simple result does not involve the Airy function or its zeros. The radiation leak of whispering gallery modes is easily obtained from this expression in (11) and the general formulas in [5]. (For a comparison see [6]).

Whispering gallery modes can be kept confined in the axial (z) direction if the rod radius is reduced to a slightly smaller radius b on both sides of the central region as shown in Fig. 1(b). The azimuthal wavenumbers of the trapped modes (with mode indices m, n in the radial and axial directions, respectively) are calculated in [7] by two different methods. First, by matching the azimuthal wavenumbers at the junction of the central region, of width $2d$ and radius a, and at the outer regions of radii b, and secondly, by a perturbation method. The results obtained from these two methods were thought to agree closely but not exactly. The result in (5) of the present short paper shows that the agreement between the two methods is, in fact, exact. The ratio of the right-hand side to the left-hand side of (5) was inaccurately given in [7] as 0.981 for the first zero (fundamental Airy mode) and 0.955 for the second zero. We now recognize that this ratio is unity for all the zeros.

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